Gauss-Codazzi equations

$(E, \bar\nabla, \langle , \rangle) = T \oplus N$ orthonormal

induces connection $\pi^T \bar\nabla = \nabla$ on T

Q: how does Ω relate to $\bar\Omega$?

In order to make this easier than last time, separate the roles of TM and E.

A: choose an o.n. frame $\overbrace{e_1 \cdots e_d}^{a}\ \overbrace{e_{d+1} \cdots e_n}^{\mu}$ for E

$\bar\nabla_i e_\alpha = e_\beta A_i{}^\beta{}_\alpha$ $\quad A_i$ skew-symmetric

$\bar\Omega = d\bar A + \bar A \wedge \bar A$

$h^\vee : N \to T$

$h : T \to N$

$$\bar A = \left[\begin{array}{c|c} A & -h^\vee \\ \hline h & * \end{array}\right]$$

$$\left[\begin{array}{c|c} A & -h^\vee \\ \hline h & * \end{array}\right]$$

$$\bar A \wedge \bar A = \left[\begin{array}{c|c} A \wedge A - h^\vee \wedge h & -A \wedge h^\vee \\ \hline h \wedge \Gamma & * \end{array}\right]$$

conclusion

1) $\bar\Omega^b{}_a = \Omega^b{}_a - h^b_\mu \wedge h^\mu_a$ $\qquad X h_a(Y) - h_b(X)\Gamma^b_a(X)$

2) $\bar\Omega^\mu{}_a = d h^\mu_a + h^\mu_b \wedge A^b{}_a$

In particular, if $S \subseteq \mathbb{R}^3$ is a surface, (e_1, e_2) on on frame,

$h^b(X) = h(X, e_b)$ is the 2^{nd} FF

1) $\Omega^1_2(X,Y) = h(e_1,X)\,h(e_2,Y) - h(e_1,Y)\,h(e_2,X)$

$= K\, d\mathrm{Vol}(X,Y)$

$X = X^1 e_1 + X^2 e_2$
$Y = Y^1 e_1 + Y^2 e_2$

2) $\nabla_X h(e_a, Y) = X(h(e_a,Y)) - h(\nabla_X e_a, Y) - h(e_a, \nabla_X Y)$

$- \Gamma^b_a(X)\, h(e_b, Y)$

$\Rightarrow \nabla h(X, e_a, Y) - \nabla h(Y, e_a, X)$

$= dh_a(X,Y) - \Gamma^b_a \wedge h_b(X,Y)$

$= 0$

$\Rightarrow \nabla h$ totally symmetric

Also on a surface, $A = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$ for some 1-form $\omega = \langle e_1, \nabla e_2\rangle$

$\Omega = d\omega$

<u>Thm</u> (Local Gauss-Bonnet) Suppose $R \subseteq (S, g)$ is a bounded region of an oriented surface S, contained in a single oriented chart $\varphi(U)$ which we take to be oriented. Suppose further that R is bounded by a single simple closed curve (equivalently, has no holes.) Then

$$\int_{\partial R} \kappa_g \, ds = 2\pi - \iint_R K \, d\sigma$$

Recall ∂R is oriented so that $\varphi^{-1}(\partial R)$ is counter-clockwise in $\mathbb{R}^2$.

Comments

- We hardly used the chart X. The assumption that R is contained in a single chart is easily eliminated (next lecture) as soon as we define integrals over general regions.

- If $S = \mathbb{R}^2$, this is the theorem of turning tangents (winding # $= 1$)

Proof Stokes thm obv. key

Cor

With R and F as above,

$$-\iint_R K\,d\sigma = \int_{\partial R} \langle De_1, e_2\rangle\,ds$$

Warning $\tilde{T}$ is not necessarily the tangent to ∂R

T

$$-\iint_R K\,d\sigma = -\iint_{x^{-1}R} K\sqrt{EG-F^2}\,du\,dv = \int_{\partial x^{-1}R} \langle \tilde{T}_u, \tilde{V}\rangle u' + \langle \tilde{T}_v, \tilde{V}\rangle v'\,ds = \int_{\partial R} \langle \tilde{T}', \tilde{V}\rangle ds$$

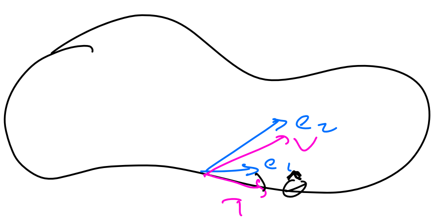
Comments

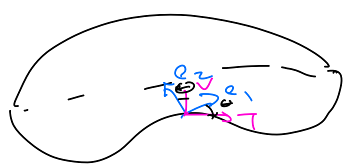
- the RHS $= \displaystyle\int_{\partial R} \langle D_T \tilde{T}, \tilde{V}\rangle\,ds$, where T is the tangent to ∂R.

  In particular, while it (seems to) depend on the frame field, it does not depend on the second fundamental form, so it is intrinsic. By the cor, the LHS is intrinsic as well. This is just Gauss's thm.

- the LHS (seems to) depend on II – though in fact it doesn't – but clearly does not depend on the choice of tangent framing $(\tilde{T}, \tilde{V})$. Hence the right hand side must not either! As long as it extends over R

In coords





If $R \subseteq$ a coord chart, Graham-Schmidt $\longrightarrow$ on. frame field (e_1, e_2)

Lem $\langle \nabla T, V \rangle - \langle \nabla e_1, e_2 \rangle = d\theta$

⌜ $T = e_1 \cos\theta + e_2 \sin\theta$

$V = -e_1 \sin\theta + e_2 \cos\theta$ ⌟

Lem $\int_{\partial R} d\theta = \int_{\partial R} d\hat{\theta}$

⌜ invariant under continuous deformation ⌟

Lem $\int_{\partial R} d\hat{\theta} = 2\pi$

⌜ Turning tangents thm ⌟

Summary:

$$\int_{\partial R} \kappa_g = 2\pi + \int \langle \nabla e_1, e_2 \rangle = 2\pi - \int_R K$$

Rmk Better picture: Ω, θ well-defined on $T^{(1)}S$.

Extension:

If ∂R is a broken geodesic, then we can use the car above to define its geodesic curvature as a <u>measure</u>, meaning we can define its integral over all sets:

$$\int_a^b K_g^{(meas)} := \Theta(b) - \Theta(a),$$

with $\Theta \mod 2\pi\mathbb{Z} = \measuredangle_{\bar{T}}^{T}$ with $\bar{T}$ parallel along the broken geodesic (cf. lec 17.2). Let $\varphi \mod 2\pi\mathbb{Z} = \measuredangle_{\tilde{T}}^{\bar{T}}$, and note

$\tilde{T} = \frac{X_u}{|X_u|}$

$$(\Theta + \varphi)(L) - (\Theta + \varphi)(0) = 2\pi$$

because a broken simple closed curve in $\mathbb{R}^2$ still has turning number 1.

Then

① $$\int_{\partial R} K_g^{(meas)} = \Theta(L) - \Theta(0) = 2\pi - (\varphi(L) - \varphi(0))$$

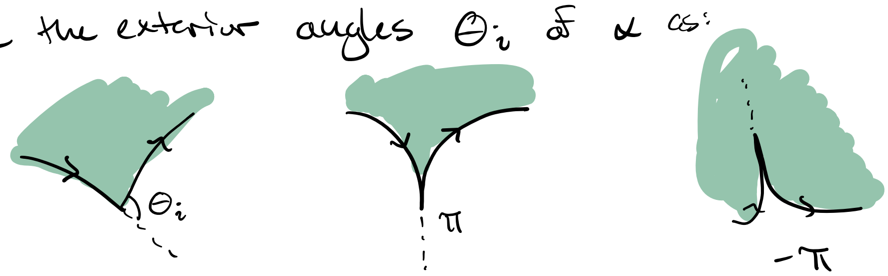
$$= 2\pi - \int_{\partial R} \varphi'$$

$$= 2\pi + \int_{\partial R} \langle \tilde{T}', \tilde{U} \rangle ds$$

$$= 2\pi - \iint_R K d\sigma$$

② $$\int_{\partial R} K_g^{(meas)} = \int_{\partial R} K_g ds + \sum_{\substack{\text{exterior} \\ \text{angles } \theta_i}} \theta_i ,$$ where we

Define the exterior angles θ_i of α as:



Putting ① + ② together gives, for broken geodesics

$$\int_{\partial R} \kappa_g \, ds + \sum_{\substack{\text{exterior} \\ \text{angles } \theta_i}} \theta_i = 2\pi - \iint_R K \, d\sigma$$

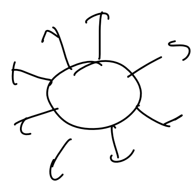
/ we only used the coord chart to produce a v.f.

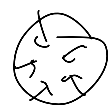
Global Gauss-Bonnet

Let X be a vector field on S with isolated zeros.

order of a zero is winding number of $\frac{X}{|X|}$ on a small circle

<u>eg</u>

1 

1 

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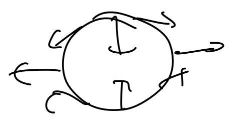
-1 

<u>Thm</u> If S is a surface w/ ∂ and X is a v.f. on S then

$$\int_{\partial S} k_g + \int_S K \overset{(+\sum_i \theta_i)}{=} 2\pi \left(\sum_p \mathrm{ord}_p(X)\right)$$

<u>Pf</u> let $\check{S} = S -$ small circle around each p_i in a chart

Use X to define e_1, e_2 on $\check{S}$

$$-\int_{\check{S}} K = \int_{\partial \check{S}} \langle \nabla e_1, e_2 \rangle = \int_{\partial S} k_g - \sum_p \overbrace{\int_{\partial D_p} \langle \nabla e_1, e_2 \rangle}^{2\pi \deg p_i}$$

Amazing fact —— LHS does not dep on X

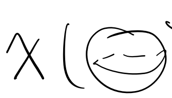
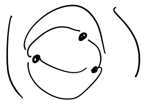
$\Longrightarrow$ invariant of S, called Euler characteristic.

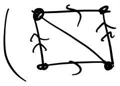
Prop In any triangulation of S,

the number of Vertices
− the number of Edges
+ the number of Faces (triangles)

is the same. It is called the Euler characteristic of S, written $\chi(S)$.

Eg $\chi(\Delta) = 3-3+1 = 1$

$\chi\Big($  $\Big) = \chi\Big($  $\Big) = 3-3+2 = 2$

$\chi\Big($  $\Big) = \chi\Big($  $\Big) = 1-3+2 = 0$

Prop Every compact oriented surface is diffeomorphic to a n-holed torus for some $n = 0,1,2,\ldots$. $\chi(n\text{-holed torus}) = 2-2n$.

$n=3$